

FUNDAMENTAL GROUPS OF RANDOM CLIQUE COMPLEXES

ABSTRACT. Clique complexes of Erdős-Rényi random graphs with edge probability between $n^{-\frac{1}{3}}$ and $n^{-\frac{1}{2}}$ are shown to be as not simply connected. This entails showing that a connected two dimensional simplicial complex for which every subcomplex has fewer than three times as many edges as vertices must have the homotopy type of a wedge of circles, two spheres and real projective planes. Note that $n^{-\frac{1}{3}}$ is a threshold for simple connectivity and $n^{-\frac{1}{2}}$ is one for vanishing first \mathbb{F}_2 homology.

1. INTRODUCTION

If n is a positive integer and $p \in [0, 1]$ is a probability write $K(n, p)$ for the probability measure on 2-dimensional simplicial complexes obtained by taking vertex set $[n] = \{1, \dots, n\}$ and edges chosen from all $\binom{n}{2}$ possibilities independently each with probability p and all triangles for which all three edges were chosen. This is the 2-skeleton of the clique complex of the Erdős-Rényi random graph. Write aas for asymptotically almost surely where the limit involved is $\lim_{n \rightarrow \infty}$.

Theorem 1.1. *For any $\epsilon > 0$ and $n^{\epsilon - \frac{1}{2}} \leq p_n \leq n^{-\epsilon - \frac{1}{3}}$ the group $\pi_1(K(n, p_n))$ is aas hyperbolic and nontrivial.*

This is proven largely by following the notation and blueprint in BHK ([1]). The main difference here is

Theorem 1.2. *If X is a finite connected two dimensional simplicial complex for which every subcomplex Y has $\frac{f^0 Y}{f^1 Y} > \frac{1}{3}$ then X has the homotopy type of a wedge of circles, two spheres and real projective planes and contains an acyclic subcomplex for which the inclusion induces an isomorphism of fundamental groups.*

Here $f^i Y$ is the number of i -dimensional faces in Y . This is a corollary of theorem 2.1 and replaces BHK Lemma 4.1 in which $\frac{f^0 Y}{f^1 Y} > \frac{1}{3}$ is replaced by $\frac{f^0 Y}{f^2 Y} > \frac{1}{2}$.

Note that if $p_n \leq n^{-1-\epsilon}$ then $K(n, p_n)$ is aas a disconnected forest and if $n^{-1+\epsilon} \leq p_n \leq n^{-\frac{1}{2}-\epsilon}$ then by Lemma 3.8 $K(n, p_n)$ is aas connected and collapsible to a graph with cycles. If $n^{-\frac{1}{3}+\epsilon} \leq p_n$ then $K(n, p_n)$ is aas simply connected from Kahle's [3] Theorem 3.4.

2. DEFINITIONS

Recall webs from BHK Definition 4.5, L from Definition 4.6 and modify Definition 4.7 to call a web W k -admissible if every $Y \subseteq W$ has $(L + k\chi)Y > 0$. Note that BHK studies 2-admissible webs and this note studies 3-admissible ones.

Theorem 2.1. *(Related to Lemma 4.16 of BHK) If W is a connected 3-admissible 2-dimensional web with $g(W) \geq 3$ then $|X|$ has the homotopy type of a wedge of circles, two spheres and real projective planes.*

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Proof:

Lemma 2.2. *If the theorem fails then there is a counterexample W with $\delta(W) \geq 2$.*

Proof: Recall from BHK the Definition 4.14 of $K(W)$ which has $\delta(K(W)) \geq 2$. $K(W)$ is also a counterexample no larger than W . \square

As in the proof of Lemma 4.22 in BHK write the admissibility sum locally as $(L + 3\chi)W = \sum_v K_v + \sum_c K_c + \sum_m K_m$ where the sums are over faces of W with empty boundary and dimensions zero, one and two respectively.

Definition 2.3. *Partially order connected two dimensional webs with $\delta \geq 2$ by setting $W \leq W'$ if*

$$\begin{aligned} & (f^2W, -(L + 3\chi)W, f^1W, f^0W, -K_{v_1}, \dots - K_{v_{f^0W}}) \\ & \leq (f^2W', -(L + 3\chi)W', f^1W', f^0W', -K_{v'_1}, \dots - K_{v'_{f^0W'}}) \end{aligned}$$

lexicographically where $v_1 \dots$ are the vertices of W with $K_{v_i} \geq K_{v_{i+1}}$ and similarly for W' .

Note that if there is a counterexample to the theorem in this partial order there is also a minimal one. Choose W to be a minimal counterexample.

Definition 2.4. *Call a two dimensional complex normal if all vertex links are connected graphs and 2-normal if all vertex links are 2-connected graphs.*

Lemma 2.5. *(Related to BHK 4.18) If W is a minimal counterexample then $\delta(W) \geq 3$ and W is 2-normal.*

Proof: If W is not normal choose $N_v(W)$ a disconnected link at v of W and consider $f : W' \rightarrow W$ the normalization map and note that every subweb $Y' \subseteq W'$ has $LfY' = LY'$ and $\chi fY' \leq \chi Y'$ so that W' is also admissible and $W' < W$ in the above order since $f^2W' = f^2W$, $LW' = LW$ and $\chi W' = \chi W - 1$. Thus some component of W' is a smaller counterexample since $|W|$ is the wedge of the components of $|W'|$ and some possibly circles.

If W is not 2-normal choose $N_v(W)$ with a cut point and consider the zipping map $z : W' \rightarrow W$ so that W' has one more vertex and one more edge than W . Note that for any $Y' \subseteq W'$ there is $LY' \geq LzY'$ and $\chi Y' \geq \chi zY'$ so that W' is also admissible and $W' < W$ in the above partial order since $f^2W' = f^2W$, $\chi W' = \chi W$, $LW' = LW + 2\mu e$. Since $|z|$ is a homotopy equivalence this makes W' a smaller counterexample.

Recall from BHK Lemma 4.11 the Definition of $C(W)$, which is defined for 2-normal webs is again a counterexample and $W \leq CW$, so $W = CW$ and hence $\delta(W) \geq 3$. \square

Lemma 2.6. *(Related to BHK 4.20) If W is a minimal counterexample then W has no monogons or digons.*

Proof: If F is a digon in W with edges e and f having $\mu e \leq \mu f$ consider the collapse W' to the shorter edge e and the collapse map $\phi : W \rightarrow W'$. Note that for every subweb $Y' \subseteq W'$ there is $Y = \phi^{-1}Y'$ and sometimes $Y_e = Y - \{e, F\}$ or $Y_f = Y - \{f, F\}$ are also subwebs. Each of these has $\chi Y_{(?) } = \chi Y'$ and at least one of $LY_{(?) } \leq LY'$ so that W' is also admissible and $W' < W$ in the above order since $f^2W' = f^2W - 1$.

If F is a monogon in W with edge e consider the collapse $\phi : W \rightarrow W'$ of F to a point. Note that for every subweb $Y' \subseteq W'$ there is $Y = \phi^{-1}Y'$ and sometimes $Y_e = Y - \{e, F\}$ is also a subweb. Both of these has $\chi_{Y(e)} = \chi_{Y'}$ and at least one of $LY_{(e)} \leq LY'$ so that W' is also admissible and $W' < W$ in the above order since $f^2W' = f^2W - 1$. \square

Lemma 2.7. *If W is a minimal counterexample with v a vertex, e and f edges containing v , c a circular 1-face, F a 2-face containing e and f and G a 2-face containing c then each of the following variables is a non negative integer:*

$$\begin{aligned} \hat{f}^1v &= f^1v - 3, \\ \hat{f}^2e &= f^2e - 3, \\ \hat{\mu}e &= \mu e - 1, \\ \hat{\chi}F &= -\chi F + 1, \\ \hat{a}(v, e, f, F) &= \hat{\mu}e + \hat{\mu}f, \\ \hat{m}(v, e, f, F) &= \mu\partial F - \hat{a}(v, e, f, F) - 3, \\ \hat{\mu}\partial(c, G) &= \mu\partial G - f^Gc\mu c \text{ and} \\ \hat{f}^Gc &= f^Gc - 1. \end{aligned}$$

Here f^im is the number of i -dimensional faces containing the face m and f^Gc is the degree of the map from the boundary of G to c . See BHK.

Note that if v is a vertex of W then using

$$\sum_{\{e|v \in e\}} \hat{\mu}e = - \sum_{\{e|v \in e\}} \frac{1}{3} \hat{\mu}e \hat{f}^2e + \sum_{\{e, f, F | v \in e \in F, v \in f \in F, e \neq f\}} \frac{1}{3} \hat{a}(v, e, f, F)$$

yields

$$\begin{aligned} & K_v \\ &= 3 - \frac{3}{2} f^1v + \sum_{\{(e, f, F) | v \in e, f \in F\}} \frac{3\chi F \frac{1}{2}(\mu e + \mu f)}{\mu\partial F} + \sum_{\{e|v \in e\}} \mu e - \sum_{\{(e, f, F) | v \in e, f \in F\}} \frac{1}{2}(\mu e + \mu f) \\ &= \frac{3}{2} - \frac{1}{2} \hat{f}^1v - \sum_{\{e|v \in e\}} \frac{1}{3} \hat{\mu}e \hat{f}^2e \\ &- \sum_{\{(e, f, F) | v \in e, f \in F\}} \frac{3\hat{\chi}F + \hat{m}(v, e, f, F) + \hat{a}(v, e, f, F) [\frac{3}{2}\hat{\chi}F + \frac{1}{6}\hat{m}(v, e, f, F) + \frac{1}{6}\hat{a}(v, e, f, F)]}{3 + \hat{m}(v, e, f, F) + \hat{a}(v, e, f, F)} \\ &= \frac{3}{2} - \frac{1}{2} \hat{f}^1v - \sum_{\{e|v \in e\}} \frac{1}{3} \hat{\mu}e \hat{f}^2e - \sum_{\{(e, f, F) | v \in e, f \in F\}} \frac{3\hat{\chi} + \hat{m} + \hat{a}[\frac{3}{2}\hat{\chi} + \frac{1}{6}\hat{m} + \frac{1}{6}\hat{a}]}{3 + \hat{m} + \hat{a}}. \end{aligned}$$

Similarly, if c is a circular one dimensional face of W then

$$K_c = \mu c \left[2 - \sum_{\{G_c \in G\}} f^Gc \frac{(f^Gc + \hat{\chi}G)\mu c + \hat{\mu}\partial(c, G)}{f^Gc\mu + \hat{\mu}\partial(c, G)} \right].$$

Finally if m is two dimensional with empty boundary then

$$K_m = 3\chi(m).$$

Since $(L + 3\chi)W > 0$ there is some face F with empty boundary and $K_F > 0$.

If m is 2 dimensional, without boundary and $K_m > 0$ then $\chi(W) > 0$ so $|W|$ is a sphere or projective plane.

Lemma 2.8. *If W is a minimal counterexample and c is a circular face then $K_c \leq 0$.*

Proof: Assume $K_c > 0$. If G contains c and $\hat{\mu}\partial(c, G) = 0$ then the contribution of G to $\frac{K_c}{\mu c}$ is $-\hat{f}^G c - \hat{\chi}G$ so only twice wrapped disks ($\hat{f} = 1, \hat{\chi} = 0$) cross caps ($\hat{f} = 0, \hat{\chi} = 1$) and singly wrapped disks ($\hat{f} = 0, \hat{\chi} = 0$) can occur if K_c is to be positive. If $\hat{\mu}\partial(c, G) \neq 0$ then $\hat{\mu}\partial(c, G) \geq g(W) \geq 3$ and $\hat{\chi}G \geq 1$ so that $\hat{f}^G c = 0$. The only faces which do not subtract at least one are the singly wrapped disks but if W is a minimal counterexample and c a circular face there is at most one of these.

This leaves only the case of one doubly wrapped and one singly wrapped disk, which has the homotopy type of a sphere and is therefore not a counterexample. \square

Lemma 2.9. *If W is a minimal counterexample and v is a vertex then $K_v \leq 0$.*

Proof: Assume that v is a vertex and $K_v > 0$.

Lemma 2.10. *(only long double edges in links) If W is a minimal counterexample, $K_v \geq K_u$ for every u adjacent to v and there are edges e and f and 2-faces F and G with F and G forming a double edge connecting e and f in the link of v then $\mu\partial F > 2(\mu e + \mu f)$ (or equivalently $\hat{m}F > \hat{\mu}e + \hat{\mu}f + 1$).*

Note that this implies that every double edge subtracts at least $\frac{4}{5}$ from K_v .

Proof: Assume not and consider $j : W'' \rightarrow W$ the deletion of G and $i : W'' \rightarrow W'$ the addition of G' which slides G across F . Note that $|W|$ and $|W'|$ are homotopy equivalent and if $G' \in Y' \subseteq W'$ then $(L + 3\chi)Y' \geq (L + 3\chi)Y$ for either $Y = Y' - G' + G$ or $Y = Y' - G' + G + F$ so that W' is admissible. The former works if e and f are in Y' , in which case $\chi Y' = \chi Y$ and $LY' = LY - \mu\partial F + 2\mu e + 2\mu f \geq LY$. Otherwise the latter works, with four cases depending on the intersection of Y' with e and f . If the intersection is empty then $LY' = LY$ and $\chi Y' = \chi Y$. If the intersection is only v then $LY' = LY$ and $\chi Y' = \chi Y + 1$. If the intersection is an edge (wlog e) then $LY' = LY + 2\mu e$ and $\chi Y' = \chi Y$. Also $X' < X$ in the above order since $f^2 X' = f^2 X$, $\chi X' = \chi X$, $LX' = LX + \mu\partial F - 2\mu e - 2\mu f \leq LX$ and $K_{v'} > K_v$. \square

Lemma 2.11. *(only long triangles in links) If W is a minimal counterexample, $K_v \geq K_u$ for every u adjacent to v and there are edges e, f and g and 2-faces E, F and G with E, F and G forming the edges and e, f and g the vertices of a triangle in the link of v then $mF + mE - 2\mu g > 2(\mu e + \mu f)$ (or equivalently $\hat{m}E + \hat{m}F > \hat{\mu}e + \hat{\mu}f + 2$).*

Note that this implies that every triangle in the link of v subtracts at least $\frac{3}{4}$ from K_v and every square with diagonal subtracts at least $\frac{6}{5}$.

Proof: Assume not and consider $j : X'' \rightarrow X$ the deletion of G and $i : X'' \rightarrow X'$ the addition of G' which slides G across E and F . Note that $|X|$ and $|X'|$ are homotopy equivalent and if $G' \in Y' \subseteq X'$ then $(L + 3\chi)Y' \geq (L + 3\chi)Y$ for $Y = Y' - G' + G$ or $Y = Y' - G' + G + F + E$. Also $X' < X$ in the above order since $f^2 X' = f^2 X$, $\chi X' = \chi X$, $LX' \geq LX$ and $K_{v'} > K_v$. \square

A case analysis now eliminates any minimal counterexample, proving Lemma 2.9. \square

This completes the proof of Theorem 2.1. \square

Proof of Theorem 1.2: The first part follows from Theorem 2.1. Since Theorem 2.1 also holds for subcomplexes the argument in the proof of Theorem 4.1 in BHK completes the proof. \square

3. FUNDAMENTAL GROUPS

The fundamental group restriction is much like in BHK.

Definition 3.1. *If X is a 2-dimensional connected simplicial complex then*

$$e_1^0 X = \min_{Y \subseteq X} \frac{f^0 Y}{f^1 Y}$$

if also X contains the vertices $\{1, \dots, w\}$ then

$$e_1^0 X_w = \min_{\{1, \dots, w\} \subseteq Y \subseteq X} \frac{f^0 Y - w}{f^1 Y}.$$

This is similar to e in BHK, but involves the ratio of vertices to edges rather than to 2-faces.

Lemma 3.2. *If X is a 2-dimensional connected simplicial complex with $e_1^0 X_w > \frac{1}{3}$ then*

$$f^1 X \leq \frac{3\chi X - 3w + LX}{3e_1^0 X_w - 1}.$$

Proof: See the proof of BHK Lemma 5.1. \square

Lemma 3.3. *For every $e > \frac{1}{3}$ there is β so that every connected 2-complex with $e_1^0(X) > e$, $L(X) \leq 0$ and $\chi(X) \leq 1$ and any contractible loop $\gamma : C_r \rightarrow X$ satisfies $A(\gamma) < \beta L(\gamma)$.*

Proof: See the proof of BHK 5.2. In this case the bound on f^1 from Lemma 3.2 replaces that on f^2 to yield only finitely many complexes to check. \square

Lemma 3.4. *For every $\epsilon > \frac{1}{3}$ there is some β so that every minimal filling $(C_r \rightarrow D \rightarrow X)$ with $e_1^0 X \geq \epsilon$ and $\chi Z \leq 1$ for every connected $Z \subseteq X$ has*

$$f^2 D < \beta(r + f^2(D - D_{\leq 0})).$$

Proof: See the proof of BHK 5.9 and the definition before 5.5 in BHK. \square

Lemma 3.5. *If $\epsilon > \frac{1}{3}$ and $(C_r \rightarrow D \rightarrow X)$ is a minimal filling with $e_1^0 X \geq \epsilon$ then*

$$f^2(D - D_{\leq 0}) \leq \frac{8r}{9\epsilon - 3}.$$

Proof: See the proof of BHK 5.10 and use $LX_{ij}^\pi = 2f^1 X_{ij}^\pi - 3f^2 X_{ij}^\pi \geq 1$ so that $(3e - 1)f^1 X_{ij}^\pi \geq \frac{3}{2}(3e - 1)f^2 X_{ij}^\pi$. \square

Lemma 3.6. *For every $\epsilon > \frac{1}{3}$ there is λ such that for every X with $e_1^0 X \geq \epsilon$, every contractible loop $\gamma : C \rightarrow X$ satisfies*

$$A\gamma \leq \lambda L\gamma.$$

Proof: See the proof of BHK 3.7. \square

Lemma 3.7. *If X has an embedded cycle $\gamma : C_6 \rightarrow X$ having $\gamma(2i) = i$ for every $i \in \{1, 2, 3\}$ and $e_1^0 X_3 > \frac{1}{3}$ then γ is not contractible in X .*

Proof: (See the proof of BHK 3.13) If $\gamma = (1, a, 2, b, 3, c)$ is a contractible cycle in X then by the second part of theorem 2.1 there is $Z \subseteq X$ such that every connected $Z' \subseteq Z$ has $\chi Z' \leq 1$ and γ is contractible in Z . Let $(C_6 \rightarrow D \rightarrow Z)$ be a minimal filling of γ in Z . By BHK Lemma 5.4, π is a 1-immersion so that no images of interior edges contribute positively to $L\gamma$ and

$$L(\text{Im}(\pi)) \leq LD \leq 6.$$

By Lemma 3.1 there is

$$f^1(\text{Im}(\pi)) \leq \frac{3\chi(\text{Im}(\pi)) - 3 \cdot 3 + L(\text{Im}(\pi))}{3e_1^0(\text{Im}(\pi))_3 - 1} \leq \frac{3 - 9 + 6}{\dots} = 0.$$

This is a contradiction and γ is not contractible in X . \square

Definition 3.8. (See BHK Definition 3.9) A 2-dimensional simplicial complex X is (ϵ_1^0, m, r) -sparse if every 2-dimensional simplicial subcomplex $Z \subseteq X$ containing the vertices $\{1, \dots, r\}$ with $f^0 Z \leq m$ satisfies $e_1^0 Z_r < \epsilon$. It is (ϵ_1^0, m, r) -full if every such complex Z occurs as a subcomplex of X .

Lemma 3.9. *If m and r are positive integers, $\epsilon > 0$ and every $p_n \leq n^{-\epsilon}$ then $K(n, p_n)$ is aas (ϵ_1^0, m, r) -sparse, while if every $p_n \geq n^{-\epsilon}$ then $K(n, p_n)$ is aas (ϵ_1^0, m, r) -full.*

Proof: See the proof of BHK 3.10 for the sparsity. Full follows from an easy second moment argument. \square

Lemma 3.10. *For every $\epsilon > \frac{1}{3}$ there are m and ρ such that every contractible loop $\gamma : C_r \rightarrow X$ in an $(\epsilon_1^0, m, 0)$ -sparse complex X satisfies $A(\gamma) < \rho L(\gamma)$.*

Proof: See the proof of the first part of Lemma 3.12 in BHK and use Lemma 3.5 in place of BHK Lemma 3.7. \square

Proof of Theorem 1.1: Since $\epsilon < \frac{1}{2}$ Lemma 3.9 with $r = 3$ implies that $K(n, p_n)$ has aas a cycle $\gamma : C_6 \rightarrow X$ with $\gamma(2i) = i$ for $i \in \{1, 2, 3\}$. By Lemma 3.6 γ is aas not contractible in X . \square

I am assured by Matthew Kahle that the arguments his paper [4] give an aas spectral gap larger than $\frac{1}{2}$ for appropriate Laplacians at all vertex links of $K(n, p_n)$ if $p_n \geq n^{-\frac{1}{2}+\epsilon}$ and that this together with a Garland type argument of Żuk imply

Theorem 3.11. *If $\epsilon > 0$ and $n^{-\frac{1}{2}+\epsilon} \leq p_n \leq n^{-\frac{1}{3}-\epsilon}$ then $\pi_1(K(n, p_n))$ aas has Kazhdan's property T.*

4. QUESTIONS

Write $K_4(n, p)$ for the measure on cell complexes given by adding a two cell to all possible cycles of length 3 and of length 4 in the Erdős-Rényi random graph $K(n, p)$ and $\pi_1(K_4(n, p))$ for the associated measure on groups.

Question: For which ϵ is $\pi_1(K_4(n, n^{-\epsilon}))$ trivial?

For this question 4-admissible webs appear to replace the 3-admissible ones arising in the clique complexes, but the local reduction methods used here do not seem to work as easily.

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